

THE TWISTOR CORRESPONDENCE OF THE DOLBEAULT COMPLEX OVER \mathbb{C}^n

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ABSTRACT. With respect to the Dolbeault complex over the flat manifold \mathbb{C}^n , an explicit description of the inverse correspondence of the twistor correspondence is given.

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Introduction

In [O.R], it is defined the twistor spaces of a Hermitian manifold. Let X be an n -dimensional Hermitian manifold and $k = 0, \dots, n$. The k -th twistor space $Z_k(X)$ of X is defined as a fiber bundle over X with fiber $G_{k,n}$. The almost complex structure of $Z_k(X)$ defined in [O.R] seems somewhat technical. In this paper, another definition of the twistor space $Z_k(X)$ is given. It is defined as an almost complex submanifold of the Riemannian twistor space $Z(X)$ by using the irreducible decomposition of the spin module Δ as a $U'(n)$ -module. ($U'(n)$ is the double covering group of $U(n)$.)

$$\Delta = \bigoplus_{k=0}^n \Delta^k.$$

In the case of Riemannian manifolds, the twistor correspondence of the spin complex (Dirac operator) is given in [M]. In the Hermitian case, it can be also defined a twistor correspondence of the twisted Dolbeault complex:

$$0 \rightarrow K^{1/2} \otimes \Lambda^{0,0} \xrightarrow{\bar{\partial}} K^{1/2} \otimes \Lambda^{0,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} K^{1/2} \otimes \Lambda^{0,n} \rightarrow 0.$$

Let H be the hyperplane bundle over $Z_k(X)$ defined by pulling back the hyperplane bundle over $Z(X)$. We assume the integrability of the almost complex structure of $Z_k(X)$. An element of the cohomology group $H^{k(n-k)}(Z_k(X), \mathcal{O}(H^{-n-1}))$ determines a k -th harmonic form of the twisted Dolbeault complex. (Theorem 1.1)

The main theme of this paper is to give the inverse correspondence explicitly in the case $X = \mathbb{C}^n$. (Definition 3.2 and Theorem 3.3) That is, to determine an element of $H^{k(n-k)}(Z_k(X), \mathcal{O}(H^{-n-1}))$ from a k -th

harmonic form of the (twisted) Dolbeault complex. It is expected to extend this result to general Hermitian manifolds.

Let us explain briefly the contents of this paper. In §1, we define twistor spaces of a Hermitian manifold by using the definition of the twistor space of a Riemannian manifold in [I]. In §2, we describe the twistor spaces of \mathbb{C}^n explicitly. In §3, we define the inverse twistor correspondence and prove the main theorem.

1. THE TWISTOR SPACES OF A HERMITIAN MANIFOLD

By a Hermitian manifold X , we mean a Riemannian manifold with a compatible almost complex structure. In this section we define twistor spaces $Z_k(X)$, $k = 0, \dots, n$ of X as submanifolds of the Riemannian twistor space $Z(X)$ defined as a submanifold of the projectivized spinor bundle $\mathbf{P}(\Delta(X))$ ([I]).

Let Δ be a spin module:

$$\Delta = \langle \theta_I \mid I < (1, \dots, n) \rangle_{\mathbb{C}}$$

where $I < (1, \dots, n)$ means that I is a subsequence of $(1, \dots, n)$. For convenience, we regard multi-indices I, J, \dots as finite sequences of possibly duplicate elements of $\{1, \dots, n\}$, and denote by IJ the composition of sequences I and J . Let us define relations among θ_I 's as:

$$\begin{aligned} \theta_{iiI} &= -\theta_I, \\ \theta_{ijI} &= -\theta_{jiI}, \quad \text{for } i \neq j. \end{aligned}$$

Then, for any multi-index I , there is a unique subsequence I_0 of $(1, \dots, n)$ such that:

$$\theta_I = \theta_{I_0} \quad \text{or} \quad \theta_I = -\theta_{I_0}.$$

Let $|I|$ denote the length of the reduced sequence I_0 , and $i \in I$ means that the number i exists in the sequence I_0 .

With this notation, we define an action of $\mathbb{R}^{2n} = \langle e_i \mid i = 1, \dots, 2n \rangle_{\mathbb{R}}$ to the spin module Δ as follows. For $i = 1, \dots, n$,

$$\begin{aligned} e_i \theta_I &= \theta_{iI} \\ e_{n+i} \theta_I &= \begin{cases} \sqrt{-1} \theta_{iI}, & \text{if } i \notin I \\ -\sqrt{-1} \theta_{iI}, & \text{if } i \in I \end{cases} \end{aligned}$$

By the definition of the Clifford algebra, this action can be extended to a $\text{CLIF}(\mathbb{R}^{2n})$ -action to Δ . Since $\mathfrak{spin}(2n)$ is a subspace of $\text{CLIF}(\mathbb{R}^{2n})$, we have a $\mathfrak{spin}(2n)$ -action on Δ . (See [I] for details.)

We identify \mathbb{R}^{2n} with \mathbb{C}^n by the complex structure defined by the matrix:

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Then we have an inclusion $\mathfrak{u}(n) \subset \mathfrak{so}(2n) = \mathfrak{spin}(2n)$. The irreducible decomposition of Δ as a $\mathfrak{u}(n)$ -module is given by:

$$\Delta = \bigoplus_{k=0}^n \Delta^k,$$

$$\Delta^k = \langle \theta_I \mid |I| = k \rangle_{\mathbb{C}}.$$

Let \mathbb{C}^n denote the vector representation of $\mathfrak{u}(n)$, then we have:

$$\Delta^k \simeq (\Lambda^k \mathbb{C}^n) \otimes (\Lambda^n \mathbb{C}^n)^{-1/2}. \quad (1.1)$$

The group $U'(n)$ corresponding to this representation is given by the following diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z}_2 & \rightarrow & \text{PIN}(2n) & \rightarrow & \text{O}(2n) \rightarrow 0 \\ & & \parallel & & \cup & & \cup \\ 0 & \rightarrow & \mathbb{Z}_2 & \rightarrow & U'(n) & \rightarrow & U(n) \rightarrow 0. \end{array}$$

Let Z denote the $\text{PIN}(2n)$ -orbit of $[\theta_0] \in \mathbf{P}(\Delta)$, which is a complex submanifold of $\mathbf{P}(\Delta)$. We define the submanifolds $Z_k, k = 0, \dots, n$ of Z by

$$Z_k = Z \cap \mathbf{P}(\Delta^k).$$

This coincides with the $U'(n)$ -orbit of $[\theta_{1\dots k}] \in \mathbf{P}(\Delta^k)$, hence by (1.1), Z_k is considered to be a Grassmannian manifold $G_{k,n}$. By the definition of Z_k , we have a diagram:

$$\begin{array}{ccc} Z & \rightarrow & \mathbf{P}(\Delta) \\ \uparrow & & \uparrow \\ Z_k & \rightarrow & \mathbf{P}(\Delta^k) \end{array} \quad (1.2)$$

The upper (lower) horizontal array is a $\text{PIN}(2n)$ - ($U'(n)$ -) equivariant mapping. We define the hyperplane bundle H over Z_k by pulling back the hyperplane bundle over Z (or equivalently $\mathbf{P}(\Delta^k)$).

Let X be a Hermitian manifold with spin structure: the structure group of X is $U'(n)$. Let P denote the principal bundle of X . Then we define the k -th twistor space and the hyperplane bundle on it as:

$$Z_k(X) = P \times_{U'(n)} Z_k$$

$$H = P \times_{U'(n)} H.$$

By (1.2), $Z_k(X)$ is a subbundle of $Z(X)$, and since Z_k is a complex submanifold of Z , it is also an almost complex submanifold. This

almost complex structure on $Z_k(X)$ is integrable if the metric of X is Bochner flat ([O.R]). In this case, the hyperplane bundle can be considered as a holomorphic line bundle.

Now we can define the twistor correspondence. Let X be a Hermitian manifold with spin structure and assume that the almost complex structure of $Z_k(X)$ is integrable. By Serre duality and (1.1), we define:

$$\begin{aligned} T : H^{k(n-k)}(Z_k(X), \mathcal{O}(H^{-n-1})) &\rightarrow \Gamma \left(X, \bigcup_{x \in X} H^{k(n-k)}(Z_k(X)_x, \mathcal{O}(H^{-n-1})) \right) \\ &= \Gamma(X, K^{1/2} \otimes \Lambda^{0,k}). \end{aligned} \quad (1.3)$$

Theorem 1.1. *An image of T is a k -th harmonic form of the twisted Dolbeault complex on X :*

$$0 \rightarrow K^{1/2} \otimes \Lambda^{0,0} \xrightarrow{\bar{\partial}} K^{1/2} \otimes \Lambda^{0,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} K^{1/2} \otimes \Lambda^{0,n} \rightarrow 0. \quad (1.4)$$

Proof. It is proved in the same way as in [H] and [M]. \square

The main result of this paper is to give an explicit description of the inverse of the above correspondence in the case $X = \mathbb{C}^n$.

2. TWISTOR SPACES OF \mathbb{C}^n

The Riemannian twistor space of \mathbb{R}^{2n} is given in [I] §8. Using that result, we give an explicit description of the twistor spaces of \mathbb{C}^n .

Let Δ' be a spin module of $\text{SPIN}(2n+2)$:

$$\Delta' = \langle \theta_I \mid I < (0, 1, \dots, n) \rangle_{\mathbb{C}}. \quad (2.1)$$

The twistor space of the $2n$ -dimensional sphere is identified with the orbit Z' of the $[\theta_\emptyset] \in \mathbf{P}(\Delta')$ under the action of $\text{PIN}(2n+2)$. Since stereographic projection defines the conformal embedding $\mathbb{R}^{2n} \subset S^{2n}$, $Z(\mathbb{R}^{2n})$ is an open submanifold of $Z(S^{2n})$. Let (Z^I) be the homogeneous coordinates with respect to (θ_I) . Then, we have:

$$Z(\mathbb{R}^{2n}) = \{(Z^I)_{I < (0, \dots, n)} \in Z' \mid \exists I < (1, \dots, n) \text{ such that } Z^I \neq 0\}.$$

Since the spin bundle of \mathbb{R}^{2n} is trivial, we have

$$Z(\mathbb{R}^{2n}) \simeq \mathbb{R}^{2n} \times Z. \quad (2.2)$$

Since a translation of \mathbb{R}^{2n} is a conformal transformation, it defines a holomorphic transformation of $Z(\mathbb{R}^{2n})$, which is representable by an element of $\text{SPIN}(2n+2; \mathbb{C})$. Let $t = (x^1, \dots, x^{2n})$ be an element of \mathbb{R}^{2n} . Then the corresponding element of $\text{SPIN}(2n+2; \mathbb{C})$ is:

$$\alpha(t) = 1 + \frac{1}{2}(x^1 e_1 + \dots + x^n e_n + x^{n+1} e_{n+2} + \dots + x^{2n} e_{2n+1})(\sqrt{-1} e_0 + e_{n+1}).$$

As an element of Z' , a point on the fiber over $0 \in \mathbb{R}^{2n}$ is written as $\sum_{I \neq 0} Z^I \theta_I$. Hence its image by the transformation $\alpha(t)$ is:

$$\alpha(t) \sum_{I \neq 0} Z^I \theta_I = Z^I \theta_I + \sqrt{-1} \sum_{k \in I} Z^{kI} \xi^k \theta_{0I} + \sqrt{-1} \sum_{k \notin I} Z^{kI} \overline{\xi^k} \theta_{0I}.$$

where $\xi^k = x^k + \sqrt{-1}x^{n+k}$ is the complex coordinate of t . It follows that the projection map to the second component of (2.2) is given by:

$$p_2 : \begin{array}{ccc} Z(\mathbb{R}^{2n}) & \rightarrow & Z \\ (Z^I)_{I < (0, \dots, n)} & \mapsto & (Z^I)_{I < (1, \dots, n)}. \end{array}$$

Hence we have:

$$Z_k(\mathbb{C}^n) = \{(Z^I) \in Z(\mathbb{R}^{2n}) \mid Z^I = 0 \text{ for all } I < (1, \dots, n) \text{ such that } |I| \neq k\}.$$

Furthermore, the coordinate transformation of (2.2) is

$$Z^{0J} = \sqrt{-1} \left(\sum_{j \in J} Z^{jJ} \xi^j + \sum_{j \notin J} Z^{jJ} \overline{\xi^j} \right), \quad \text{for } 0 \notin J. \quad (2.3)$$

Hence the horizontal $(1, 0)$ -forms of $Z_k(\mathbb{C}^n)$ is given by the following proposition.

Proposition 2.1. *Horizontal $(1, 0)$ -forms on $Z_k(\mathbb{C}^n) \simeq \mathbb{C}^n \times Z_k$ at $(\xi) \times (Z^I)$ is spanned by*

$$\begin{aligned} \sum_{j \notin J} Z^{jJ} \overline{d\xi^j}, \quad |J| = k - 1, \\ \sum_{j \in J} Z^{jJ} d\xi^j, \quad |J| = k + 1. \end{aligned}$$

Proof. A point z of the space Z_k represents a complex structure of \mathbb{R}^{2n} . Let us compare it with the original one corresponding to the point $z_0 = [\theta_0] \in Z$. For convenience, we call $(1, 0)_z$ -forms the $(1, 0)$ -forms with respect to the complex structure corresponding to z . Put $z_1 = \theta_{(12 \dots k)}$. Then $(1, 0)_{z_1}$ -forms are spanned by $\overline{d\xi^1}, \dots, \overline{d\xi^k}, d\xi^{k+1}, \dots, d\xi^n$. This mean that the complexified form ω is of type $(1, 0)_{z_1}$ if and only if the both $(1, 0)_{z_0}$ -part and $(0, 1)_{z_0}$ -part of ω are of type $(1, 0)_{z_1}$. Hence the proposition follows immediately from (2.3). \square

3. MAIN THEOREM

By restricting the $\text{SO}(2n)$ -action on Z , we have a $\text{U}(n)$ -action on Z_k . This action can be complexified and we have a $\text{GL}(n; \mathbb{C})$ -action on Z_k . This action defines a \mathbb{C} -linear mapping:

$$\mathcal{F} : \mathfrak{gl}(n; \mathbb{C}) \rightarrow \Gamma(Z_k, \Theta),$$

where Θ is the holomorphic tangent bundle on Z_k , which is considered to be the $(1, 0)$ -part of the complexified tangent bundle $TZ_k \otimes \mathbb{C}$. With respect to Lie algebra structures, we have:

$$\mathcal{F}([a, b]) = -[\mathcal{F}(a), \mathcal{F}(b)]. \quad (3.1)$$

Let (E^j_i) denote a standard basis of $\mathfrak{gl}(n; \mathbb{C})$. We define a vector field \mathcal{F}^j_i to be:

$$\mathcal{F}^j_i = -\mathcal{F}({}^t E^j_i).$$

We define operators acting on the differential forms on $Z_k(\mathbb{C}^n)$:

$$\begin{aligned} D_\alpha &= \sum \text{ext}(d\xi^a) i(\overline{\mathcal{F}^a_b}) L_{\frac{\partial}{\partial \xi^b}}, \\ D_\beta &= \sum L_{\frac{\partial}{\partial \xi^a}} i(\overline{\mathcal{F}^a_b}) \text{ext}(\overline{d\xi^b}), \end{aligned}$$

where ext and i denote the exterior and inner multiplication, and L denotes the Lie derivative.

Let F be a power series defined as:

$$F(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}$$

This function and its derivatives play an important role by the following property.

Lemma 3.1. *Let l be a non-negative integer. We have an equality*

$$x F^{(l+2)}(x) + (l+1) F^{(l+1)}(x) - F^{(l)}(x) = 0.$$

Let $\Lambda_V^{0,k(n-k)}$ denote the line subbundle of $\Lambda^{0,k(n-k)} Z_k(\mathbb{C}^n)$ spanned by vertical forms. If we identify H^{-1} with \overline{H} by the standard Hermitian metric, we have

$$\Lambda^{0,k(n-k)} \otimes H^{-n-1} \supset \Lambda_V^{0,k(n-k)} \otimes H^{-n-1} \simeq \overline{H}$$

since we have $\Lambda_V^{0,k(n-k)} \simeq \overline{H}^{-n}$. On the other hand, we have an isomorphism $\Gamma(Z_k, \mathcal{O}(H)) \simeq K^{-1/2} \otimes \Lambda^{k,0}$. Hence we have a natural map:

$$j : \Gamma(\mathbb{C}^n, K^{1/2} \otimes \Lambda^{0,k}) \rightarrow \Gamma(Z_k(\mathbb{C}^n), \Lambda_V^{0,k(n-k)} \otimes H^{-n-1})$$

Definition 3.2. We define an operator:

$$\begin{aligned} \mathcal{A} : \Gamma(\mathbb{C}^n, K^{1/2} \otimes \Lambda^{0,k}) &\rightarrow \Gamma(Z_k(\mathbb{C}^n), \Lambda^{0,k(n-k)} \otimes H^{-n-1}) \\ a &\mapsto k!(n-k)! F^{(k)}(D_\beta) F^{(n-k)}(D_\alpha) j(a). \end{aligned}$$

Now we can state the main theorem.

Theorem 3.3. *Let f be a twisted $(0, k)$ -form on \mathbb{C}^n . Then $\mathcal{A}(f)$ is $\bar{\partial}$ -closed if and only if f is a harmonic form with respect to the twisted Dolbeault complex (1.4). The restriction of \mathcal{A} to the space of harmonic forms is the inverse correspondence of T of (1.3)*

With respect to the standard complex structure of \mathbb{C}^n , we have a decomposition of the complexified horizontal cotangent bundle:

$$\Lambda_H^1 \otimes \mathbb{C} = \Lambda_H^{(1,0)} \oplus \Lambda_H^{(0,1)},$$

By this decomposition, we define two projections as follows:

$$\pi_\gamma : \Lambda^1 \rightarrow \begin{cases} \Lambda_H^{(1,0)}, & \gamma = \alpha, \\ \Lambda_H^{(0,1)}, & \gamma = \beta. \end{cases}$$

Let γ be α or β . We define:

$$d_\gamma = \pi_\gamma \circ d.$$

Lemma 3.4. 1. *Let γ be α or β , and put $E_\gamma = [d, D_\gamma]$. Then, we have*

$$\begin{aligned} E_\alpha &= -\text{ext}(d\xi^a) L_{\overline{\mathcal{F}}_b^a} L_{\frac{\partial}{\partial \xi^b}}, \\ E_\beta &= L_{\frac{\partial}{\partial \xi^a}} L_{\overline{\mathcal{F}}_b^a} \text{ext}(\overline{d\xi^b}). \end{aligned}$$

2. *Let f be a power series and γ be α or β , then*

$$[d, f(D_\gamma)] = f'(D_\gamma)E_\gamma - f''(D_\gamma)D_\gamma d_\gamma.$$

3. *Put $\Gamma = [E_\beta, D_\alpha]$. Then, we have*

$$\begin{aligned} \Gamma &= \text{ext}(d\xi^a) i(\overline{\mathcal{F}}_b^a) \text{ext}(\overline{d\xi^b}) L_{\frac{\partial}{\partial \xi^c}} L_{\frac{\partial}{\partial \xi^c}} + L_{\frac{\partial}{\partial \xi^a}} i(\overline{\mathcal{F}}_b^a) L_{\frac{\partial}{\partial \xi^b}} \text{ext}(d\xi^c \wedge \overline{d\xi^c}), \\ \Gamma D_\alpha &= D_\alpha \Gamma, \\ [E_\beta, f(D_\alpha)] &= f'(D_\alpha) \Gamma. \end{aligned}$$

Proof. (1) and (3) are proved by simple computation.

(2) It suffices to prove

$$[d, D_\gamma^n] = nD_\gamma^{n-1}E_\gamma - (n-1)nD_\gamma^{n-1}d_\gamma,$$

which can be proved by induction on n by using the formula:

$$[E_\gamma, D_\gamma] = -2D_\gamma d_\gamma. \quad (3.2)$$

By (1), we have:

$$[E_\alpha, D_\alpha] = -\text{ext}(d\xi^a \wedge d\xi^c) i([\overline{\mathcal{F}}_b^a, \overline{\mathcal{F}}_d^c]) L_{\frac{\partial}{\partial \xi^b}} L_{\frac{\partial}{\partial \xi^d}}$$

Since, by (3.1), we have:

$$[\mathcal{F}_b^a, \mathcal{F}_d^c] = \delta_d^a \mathcal{F}_b^c - \delta_b^c \mathcal{F}_d^a$$

hence we have shown (3.2) when $\gamma = \alpha$. The case $\gamma = \beta$ can be proved in a similar way. \square

By Lemma 3.1 and Lemma 3.4, we have:

$$\begin{aligned} dF^{(k)}(D_\beta)F^{(n-k)}(D_\alpha) &= F^{(k)}(D_\beta)F^{(n-k)}(D_\alpha)(d - d_\alpha - d_\beta) \\ &\quad + F^{(k)}(D_\beta)F^{(n-k+1)}(D_\alpha)\{E_\alpha + (n - k + 1)d_\alpha\} \\ &\quad + F^{(k+1)}(D_\beta)F^{(n-k)}(D_\alpha)\{E_\beta + (k + 1)d_\beta\} \\ &\quad - F^{(k+1)}(D_\beta)F^{(n-k+1)}(D_\alpha)\Gamma \end{aligned}$$

Since $j(f)$ is harmonic in the vertical direction,

$$F^{(k)}(D_\beta)F^{(n-k)}(D_\alpha)(d - d_\alpha - d_\beta)j(f) = 0,$$

and by Lemma 3.4 (3), if f is harmonic, we have

$$\Gamma j(f) \equiv 0 \quad \text{modulo } (1, 0)\text{-forms.}$$

To compute the action of E_α and E_β , we have to take local coordinates of Z_k . Let I be a subsequence of $(1, \dots, n)$ of length k . Then

$$w_{ij} = Z^{ijI}/Z^I, \quad i \in I, j \notin I$$

are local coordinates of $U_I = \{(Z^J)_{J < (1 \dots n)} \in Z_k \mid Z^I \neq 0\}$. Put $z^J = Z^J/Z^I$ for $J < (1, \dots, n)$.

Lemma 3.5. *The vector field \mathcal{F}^a_b is written in the local coordinates as:*

$$\mathcal{F}^a_b = - \sum_{\substack{i \in I \\ j \notin I}} z^{aiI} z^{bjI} \frac{\partial}{\partial w_{ij}}.$$

Let δ be a function:

$$\delta(a) = \begin{cases} 1, & \text{if } a \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma is a translation of the Plücker relation.

Lemma 3.6. 1. *We fix a multi-index I and local coordinates w_{ij} as above.*

$$\frac{\partial z^J}{\partial w_{ij}} = \begin{cases} -z^{ijJ} & \text{if } i \notin J, j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

2. *Let J, K be multi-indices of length $|J| = k + 1, |K| = k - 1$.*

$$\sum_{a \in J \setminus K} Z^{aJ} Z^{aK} = 0$$

3. Let J, K be multi-indices of length $|J| = |K| = k$.

$$\sum_{j \in J \setminus K} Z^{ajJ} Z^{bjK} = -\delta(a \notin K) Z^J Z^{baK} + \delta(a \notin J) Z^{baJ} Z^K.$$

Proof. (1) It is obvious if $|J \setminus I| \leq 1$. If $|J \setminus I| \geq 2$, let $i_k \in I \setminus J$, $j_k \in J \setminus I$, $k = 1, 2$, be numbers such that $i_1 \neq i_2$, $j_1 \neq j_2$. Then, by [1] Corollary 3.3, we have

$$Z^J = \frac{1}{Z^{J i_1 i_2 j_1 j_2}} (-Z^{J i_2 j_2} Z^{J i_1 j_1} + Z^{J i_2 j_1} Z^{J i_1 j_2}).$$

Hence we complete the proof by induction.

(2) This follows by induction on $|J \setminus K|$ by using (1).

(3) We prove the case in which $a \neq b$. Put

$$\begin{aligned} A &= \sum_{j \in J \setminus K} Z^{ajJ} Z^{bjK} \\ &= \sum_{j \in J \setminus K} Z^{jaJ} Z^{jbK} - 2\delta(a \in J \setminus K) Z^J Z^{baK} - 2\delta(b \in J \setminus K) Z^{abJ} Z^K. \end{aligned}$$

Since

$$\begin{aligned} P &= (J \setminus K) \setminus (aJ \setminus bK) = \delta(a \in J \setminus K) a \cup \delta(b \in J \setminus K) b, \\ M &= (aJ \setminus bK) \setminus (J \setminus K) = \delta(a \notin J \cup K) a \cup \delta(b \in J \cap K) b, \end{aligned}$$

we have:

$$\begin{aligned} A &= \sum_{j \in aJ \setminus bK} Z^{jaJ} Z^{jbK} + \sum_{j \in P} Z^{jaJ} Z^{jbK} - \sum_{j \in M} Z^{jaJ} Z^{jbK} - 2 \sum_{j \in P} Z^{jaJ} Z^{jbK} \\ &= - \sum_{j \in P \cup M} Z^{jaJ} Z^{jbK} \quad \text{by (2)} \\ &= -\delta(a \notin K) Z^J Z^{baK} + \delta(a \notin J) Z^{baJ} Z^K. \end{aligned}$$

The case in which $a = b$ can be proved in the same way. \square

Lemma 3.7. Let I be a subsequence of $(1, \dots, n)$ and $\overline{s^I}$ be the image of $\Lambda_{i \in I} \overline{d\xi^i}$ by j .

$$L_{\overline{\mathcal{F}}_a} \overline{s^I} = -(k+1) \delta_b^a \overline{s^I} - \delta(a \notin I) \overline{s^{baI}} - (n+1) \sum_{J \ni a} \frac{z^{baJ} \overline{z^J}}{N} \overline{s^I} \quad (3.3)$$

$$L_{\overline{\mathcal{F}}_a} \overline{s^I} = (n-k+1) \delta_b^a \overline{s^I} + \delta(b \in I) \overline{s^{abI}} + (n+1) \sum_{J \not\ni b} \frac{z^{abJ} \overline{z^J}}{N} \overline{s^I} \quad (3.4)$$

where

$$N = \sum_J \left| \frac{z^J}{z^I} \right|^2$$

Proof. Let w_{ij} be the local coordinates as above. Let $\overline{\rho^I}$ be the image of the standard trivialization of \overline{H} over $\mathbb{C}^n \times U_I$ by the isomorphism $\overline{H} \simeq H^{-1}$. Let $\overline{K^I}$ be the standard trivialization of $\Lambda_V^{0,k(n-k)}$ over $\mathbb{C}^n \times U_I$. Then

$$\overline{s^I} = \overline{\rho^I}^{\otimes(n+1)} \otimes \overline{K^I}.$$

First, we compute:

$$\begin{aligned} L_{\overline{\mathcal{F}^a_b}} \overline{\rho^I} &= \nabla_{\overline{\mathcal{F}^a_b}} \overline{\rho^I} \\ &= -\frac{\overline{\mathcal{F}^a_b}(N)}{N} \overline{\rho^I} \\ &= \sum_{\substack{i \in I \\ j \notin I}} \frac{z^{aiI} z^{bjI} \frac{\partial}{\partial w_{ij}} N}{N} \overline{\rho^I} \quad [\text{By Lemma 3.5}] \\ &= -\sum_J \sum_{\substack{i \in I \setminus J \\ j \in J \setminus I}} \frac{z^J \overline{z^{ijJ} z^{aiI} z^{bjI}}}{N} \overline{\rho^I} \quad [\text{By Lemma 3.6 (1)}] \\ &= -\sum_J \sum_{i \in I \setminus J} \frac{z^J \overline{z^{aiI} z^{biJ}}}{N} \overline{\rho^I} \quad [\text{By Lemma 3.6 (3)}] \\ &= \left(\sum_{J \ni a} \frac{z^J \overline{z^{baJ}}}{N} - \delta(a \notin I) \overline{z^{baI}} \right) \overline{\rho^I} \quad [\text{By Lemma 3.6 (3)}] \end{aligned}$$

By changing indices we have:

$$L_{\overline{\mathcal{F}^a_b}} \overline{\rho^I} = \begin{cases} (-\delta_b^a - \delta(a \notin I) \overline{z^{baI}} - \sum_{J \ni a} \frac{z^{baJ} \overline{z^J}}{N}) \overline{\rho^I}, & \text{for (3.3)} \\ (\delta_b^a + \delta(b \in I) \overline{z^{abI}} + \sum_{J \ni b} \frac{z^{abJ} \overline{z^J}}{N}) \overline{\rho^I}, & \text{for (3.4)} \end{cases} \quad (3.5)$$

Second, we compute:

$$L_{\overline{\mathcal{F}^a_b}} d\overline{w_{ij}} = d\overline{\mathcal{F}^a_b(w_{ij})} = (-\delta_b^i \overline{z^{aiI}} + \delta_j^a \overline{z^{bjI}}) d\overline{w_{ij}} + \dots$$

Hence:

$$\begin{aligned} L_{\overline{\mathcal{F}^a_b}} \overline{K^I} &= \sum_{\substack{i \in I \\ j \notin J}} (-\delta_b^i \overline{z^{aiI}} + \delta_j^a \overline{z^{bjI}}) \overline{K^I} \\ &= \{-(n-k)\delta(b \in I) \overline{z^{abI}} + k\delta(a \notin I) \overline{z^{baI}}\} \overline{K^I}. \end{aligned}$$

By changing indices we have:

$$L_{\overline{\mathcal{F}^a_b}} \overline{K^I} = \begin{cases} \{(n-k)\delta_b^a + n\delta(a \notin I) \overline{z^{baI}}\} \overline{K^I}, & \text{for (3.3),} \\ \{-k\delta_b^a - n\delta(b \in I) \overline{z^{abI}}\} \overline{K^I}, & \text{for (3.4).} \end{cases} \quad (3.6)$$

Hence, by (3.5) and (3.6), we complete the proof. \square

The next lemma follows immediately from Proposition 2.1 and Lemma 3.7.

Lemma 3.8. *Let $f_{\overline{I}}\overline{s^I}$ be an image of \mathcal{A} . Then we have*

$$\begin{aligned} \{E_\beta + (k+1)d_\beta\}f_{\overline{I}}\overline{s^I} &\equiv - \sum_{a \notin I} \frac{\partial f_{\overline{I}}}{\partial \xi^a} \overline{d\xi^b} \wedge \overline{s^{baI}}, \\ \{E_\alpha + (n-k+1)d_\alpha\}f_{\overline{I}}\overline{s^I} &\equiv - \sum_{b \in I} \frac{\partial f_{\overline{I}}}{\partial \xi^b} d\xi^a \wedge \overline{s^{abI}}, \end{aligned}$$

where \equiv denotes the equivalence modulo $(1,0)$ -forms.

This lemma shows that the coefficient of $\overline{d\xi^b s^I}$ ($d\xi^a \overline{s^I}$) in $\mathcal{A}(f)$ is equal to the coefficient of $\overline{s^{bI}}$ ($\overline{s^{aI}}$) in the image of the Dirac operator. Therefore, we complete the proof of the theorem.

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